JOURNAL OF COMPUTATIONAL PHYSICS 69, 49–68 (1987)

A Method for Computing Flow Fields around Moving Bodies

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Received April 24, 1985; revised March 5, 1986

A new method for computing flow fields with arbitrarily moving boundaries is proposed. Under the concept of Lie derivatives the field equations in general moving coordinates are derived, which consist of several kinds of equations, for example, one written in Viviand's conservative form. According to our formulation, it is natural and reasonable to consider that the computational coordinates fitted to the body move in space, contrary to the usual computational procedures. The two-dimensional incompressible Navier–Stokes equations in general moving coordinates are solved by a finite difference method. The present calculations are made for (a) the blood flow in human ventricle, and (b) the dynamic stall process on oscillating airfoil. Consequently it is shown that the flows generated by moving bodies can easily be analyzed by the present method. © 1987 Academic Press, Inc.

1. INTRODUCTION

The recent progresses of numerical analysis and computer performance have made it possible to numerically calculate unsteady flow problems by directly solving the Navier–Stokes equations with a large number of grid points within reasonable computation time. The flow fields around arbitrarily moving boundaries, however, can scarcely have been analyzed among the phenomena usually observed in nature. To analyze such problems, Viviand [1] derived the conservative forms in the moving coordinates, and Lerat and Sides [2] and Steger [3] used the equations to calculate flow around the oscillating airfoil. In process of derivation of equations in the moving coordinates from the equations in Cartesian coordinates, Viviand used the transformation of variables. Since the procedure of derivation is very complicated and is not straightforward, and it is rather difficult to grasp the meaning of the equations.

In this paper we propose a new method for computing flow fields with arbitrarily moving boundaries. It is shown that the field equations in general moving coordinates can be derived under the concept of Lie derivative [4, 5] without labour of complexity. They are obtained straightforward if the tensorial character of the equation is known. In our formulation the grid fitted to the body moves and is not fixed in space, contrary to the usual computation procedures. The main purpose of

0021-9991/87 $3.00

this article is to examine whether reasonable solutions will be obtained by using the present formulation or not, and the two-dimensional incompressible Navier–Stokes equations in the moving general coordinates are solved for two different flows. The first is the numerical simulations of the blood flow in human left ventricle by using $\psi - \omega$ variables, and the second is the numerical simulation of the dynamic stall process of the NACA0012 airfoil by using P-$\mathbf{v}$ variables. In each problem it is shown that the unsteady flow problems generated by the motion of bodies can easily be solved by using our formulation.

2. Governing Equations in Moving Coordinates

In general coordinates fixed in space, it is well known that the equation of conservation law is expressed as follows:

$$\partial\tilde{f}/\partial t + \partial_k(\tilde{f}v^k - \tilde{j}^k) = \tilde{\gamma},$$

where $\tilde{f}$, $\tilde{f}v^i$, $\tilde{j}^i$, and $\tilde{\gamma}$ are the density distribution of physical quantities in a unit cell of the coordinate system, the convective flux density, the conductive flux density, and the source density, respectively. These elements of conservation equation are the tensor density of weight $+1$, and the spacial derivative ∂_k must be replaced by the covariant derivative ∇_k in index notation.

The equations of mass and momentum are written as follows:

$$\partial\tilde{\rho}/\partial t + \nabla_k(\tilde{\rho}v^k) = 0, \tag{1}$$

$$\partial\tilde{\rho}v^i/\partial t + \nabla_k(\tilde{\rho}v^i v^k - \tilde{\sigma}^{ik}) = \tilde{\rho}F^i, \tag{2}$$

where ρ is the mass density and σ^{ik} the stress tensor, and F^i the body force, and the notation of summation convention is used. The upper symbol ($\sim$) denotes the quantity multiplied by $\sqrt{g} = \mathrm{Det}(g_{ij})^{1/2}$, which is the scalar density of weight $+1$, and g_{ij} is the metric tensor.

Now, let us derive the equations in general moving coordinates from Eqs. (1) and (2). Let the general moving coordinates $\mathbf{x}_*$ be denoted by $\mathbf{x}$ at time $t = t_0$, namely, $\mathbf{x}_*$ satisfies

$$\mathbf{x}_* = \mathbf{x}_*(\mathbf{x}, t - t_0), \qquad \mathbf{x}_*(\mathbf{x}, 0) = \mathbf{x},$$

then it is obvious that the spacial derivatives satisfy $\nabla_{i*} = \nabla_i$ at $t = t_0$. On the other hand the time derivative $\partial/\partial t$ in Eqs. (1) and (2) is the evaluation at a fixed point in space, and must be rewritten by the evaluation at the same coordinates $\mathbf{x}_*$: $\partial/\partial t|\mathbf{x}_*$. Such time derivative is named as the Lie derivative [4, 5] (see Appendix, (i)). Applying the concept of Lie derivative, the equations in arbitrarily moving coordinates are straightforward obtained. Since $\tilde{\rho}$ and $\tilde{\rho}v^i$ are the scalar density of weight $+1$ and the contravariant vector density of weight $+1$, respectively, Eqs. (1) and (2) are rewritten in moving coordinates as follows using Eqs. (A3) and (A5):

$$\partial\tilde{\rho}/\partial t + \nabla_k[\tilde{\rho}(v^k - V^k)] = 0, \tag{3}$$

$$\partial(\tilde{\rho}v^i)/\partial t + \nabla_k[\tilde{\rho}v^i(v^k - V^k) - \tilde{\sigma}^{ik}] + \tilde{\rho}v^k \nabla_k V^i = \tilde{\rho}F^i, \tag{4}$$

where the subscript $_*$ is omitted for simplicity, and V^i is the velocity field of the moving coordinates. In the case that the moving velocities of the coordinate system coincide with the velocities of fluid ($v^i = V^i$), the coordinate system is termed the Lagrange coordinates. In this case, Eqs. (3) and (4) become

$$\partial\tilde{\rho}/\partial t = 0. \tag{5}$$

$$\tilde{\rho}(\partial v^i/\partial t + v^k \nabla_k v^i) = \nabla_k \tilde{\sigma}^{ik} + \tilde{\rho}F^i \tag{6}$$

which correspond to the usual expressions in the mechanics of solid [6]. Also, if the x, y, z, components of $\mathbf{v}$ are used,

$$\mathbf{v} = v^i\mathbf{e}_i = v^m\mathbf{i}_m \qquad (m = x, y, z),$$

where $\mathbf{i}_m$ is the rectangular base vector, then Viviand's conservative forms are easily derived from Eqs. (3) and (1) or Eq. (4) as follows [see Appendix, (ii)],

$$\partial\tilde{\rho}/\partial t + \partial_k[\tilde{\rho}(v^k - V^k)] = 0 \tag{7}$$

$$\partial(\tilde{\rho}v^m)\,\partial t + \partial_k[\tilde{\rho}v^m(v^k - V^k) - \tilde{\sigma}^{mk}] = \tilde{\rho}F^m \qquad (m = x, y, z). \tag{8}$$

If the tensorial character of the equation is known, then the field equation in general moving coordinates can be derived in the same way without labour of complexity.

Now, let us derive the incompressible Navier–Stokes equations in general moving coordinates. The equation of mass conservation is

$$\nabla_k \tilde{v}^k = \partial_k \tilde{v}^k = 0, \tag{9}$$

where the first equality is due to $\tilde{v}^i$ be the contravariant vector density of weight $+1$. Using the stress tensor of Newtonian fluid [7],

$$\sigma^{ij} = -Pg^{ij} + \zeta g^{ij}\nabla_k v^k + \mu[\nabla^j v^i + \nabla^i v^j - \tfrac{2}{3}g^{ij}\nabla_k v^k], \tag{10}$$

where P is the pressure, and ζ and μ are the bulk viscosity and the coefficient of viscosity, respectively, the equation of momentum conservation is obtained as follows,

$$\partial\tilde{v}^i/\partial t + (1/\sqrt{g})\,\tilde{v}^k \nabla_k \tilde{v}^i = g^{ik}\nabla_k(-\tilde{P}/\rho) + \nu\nabla_k(g^{km}\nabla_m \tilde{v}^i), \tag{11}$$

where $\nu = \mu/\rho$, and g^{ij} is the contravariant metric tensor. In moving coordinate system the momentum conservation equation is written as

$$\begin{aligned}\partial\tilde{v}^i/\partial t &+ (1/\sqrt{g})[(\tilde{v}^k - \tilde{V}^k)\nabla_k \tilde{v}^i - \tilde{v}^i\,\partial_k \tilde{V}^k + \tilde{v}^k \nabla_k \tilde{V}^i] \\ &= g^{ik}\nabla_k(-\tilde{P}/\rho) + \nu\nabla_k(g^{km}\nabla_m \tilde{v}^i),\end{aligned} \tag{12}$$

where Eq. (A5) is used since $\tilde{v}^i$ is the contravariant vector density of weight +1. In two-dimensional problems, the $\psi-\omega$ method is known to be efficient in computation, and is used in several problems [8]. Here, we also derive the equations of ψ and ω in general moving coordinate system. The vortex ω_j is defined by $\omega_j = \tilde{\varepsilon}_{jni} g^{nh} \nabla_h \tilde{v}^i$, and the equation of ω_j is obtained by operating $\tilde{\varepsilon}_{jni} g^{nh} \nabla_h$ to Eq. (11) as

$$\partial\omega_j/\partial t + (1/\sqrt{g})(\tilde{v}^k \nabla_k \omega_j + \tilde{\varepsilon}_{jni} g^{nh} \nabla_h \tilde{v}^k \nabla_k \tilde{v}^i) \\ = \nu \nabla_k (g^{km} \nabla_m \omega_j). \tag{13}$$

Note that the permutation symbol $\tilde{\varepsilon}_{ijk}$ is the tensor density of weight −1, and the vortex ω_j is the covariant vector. The expression in moving coordinates is obtained by using Eq. (A6) as follows:

$$\partial\omega_j/\partial t + (1/\sqrt{g})[(\tilde{v}^k - \tilde{V}^k) \nabla_k \omega_j - \omega_k \nabla_j \tilde{V}^k + \varepsilon_{jni} g^{nh} \nabla_h \tilde{v}^k \nabla_k \tilde{v}^i) \\ = \nu \nabla_k (g^{km} \nabla_m \omega_j). \tag{14}$$

In two-dimensional case, the metric tensor g_{ij} is given by

$$g_{ij} = \begin{vmatrix} g_{11}, & g_{12}, & 0 \\ g_{21}, & g_{22}, & 0 \\ 0, & 0, & 1 \end{vmatrix}$$

and vortex ω_j has only one component ω_3, and $\omega_1 = \omega_2 = 0$. Hereafter the index of ω is omitted for simplicity. The stream function ψ is defined as

$$\tilde{v}^1 = \partial\psi/\partial x^2, \qquad \tilde{v}^2 = -\partial\psi/\partial x^1, \tag{15}$$

then Eq. (9) is satisfied automatically. After simple calculations, the equations of $\psi-\omega$ method in general moving coordinates are obtained as follows:

$$\partial\omega/\partial t + (1/\sqrt{g})[(\partial\psi/\partial x^2 - \tilde{V}^1)\, \partial\omega/\partial x^1 \\ -(\partial\psi/\partial x^1 + \tilde{V}^2)\, \partial\omega/\partial x^2] = \nu \Lambda \cdot \omega, \tag{16}$$

$$-\omega = \Lambda \cdot \psi, \tag{17}$$

where $\Lambda \cdot \equiv A\, \partial_1 \partial_1 + B\, \partial_1 \partial_2 + C\, \partial_2 \partial_2 + D_1 \partial_1 + D_2 \partial_2$, and the constants are defined as follows using the Riemann connection $\{{}_j{}^i{}_k\}$,

$$A \equiv g^{11},\ B = 2g^{12},\ C \equiv g^{22},\ D_S \equiv -(A\{{}_1{}^S{}_1\} + B\{{}_1{}^S{}_2\} + C\{{}_2{}^S{}_2\}) \quad (s = 1, 2).$$

After all the incompressible Navier–Stokes equations in general moving coordinates are given by Eqs. (9) and (12) for $P-\mathbf{v}$ variables, and by Eqs. (16) and (17) for $\psi-\omega$ variables, respectively.

3. Computational Scheme

In this paper two-dimensional incompressible flows are solved for two cases: (1) the blood flow in human ventricle by using $\psi-\omega$ variables and (2) the dynamic stall process on oscillating airfoil by $P-\mathbf{v}$ variables. The computational schemes in both cases are similar, and for the computations by $\psi-\omega$ variables see [8], and for the computations by $P-\mathbf{v}$ variables see [9] which is the modifications of MAC method [10]. In both cases, we must solve the Poisson equation and the evolution equation.

The Poisson equation of ψ or P is solved by the SLOR method, and the SLOR scheme is written for ψ as

$$\begin{aligned}&\mathbf{A}(\psi^+_{I+1,J}-2\psi^+_{I,J}+\psi_{I-1,J})+\mathbf{B}(\psi_{I+1,J+1}+\psi_{I-1,J-1}-\psi_{I-1,J+1}-\psi_{I+1,J-1})\\&\quad+\mathbf{C}[\psi_{I,J+1}-(2/\Omega)\,\psi^+_{I,J}-2(1-1/\Omega)\,\psi_{I,J}+\psi_{I,J-1}]\\&\quad+\mathbf{D}_1(\psi_{I+1,J}-\psi_{I-1,J})+\mathbf{D}_2(\psi_{I,J+1}-\psi_{I,J-1})=-\omega^n_{I,J},\end{aligned}\tag{18}$$

where the suffix $(+)$, (n), and (I,J) denote the value of the present iteration, the time step, and the grid point, respectively. Ω is the relaxation parameter, and we take $\Omega\sim 1.4$ from the numerical experiments. The constants $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}_1$ and $\mathbf{D}_2$ are defined by

$$\mathbf{A}=g^{11}/(\Delta x^1)^2,\qquad \mathbf{B}\equiv 2g^{12}/(\Delta x^1\,\Delta x^2),\qquad \mathbf{C}\equiv g^{22}/(\Delta x^2)^2,$$

$$\mathbf{D}_S=-\left(g^{11}\left\{{S\atop 1\;1}\right\}+2g^{12}\left\{{S\atop 1\;2}\right\}+g^{22}\left\{{S\atop 2\;2}\right\}\right)/(2\,\Delta x^S)\quad (s=1,2).$$

The equation of pressure P is the same with Eq. (18) except that the r.h.s. term is written as

$$\begin{aligned}&\text{r.h.s.}=-(\nabla_j\tilde{v}^k\,\nabla_k\tilde{v}^j)/g-\partial D/\partial t-v^k\,\partial_k D+\nu g^{kj}\,\nabla_k(\partial_j D),\\&D\equiv\partial_k\tilde{v}^k/\sqrt{g},\end{aligned}\tag{19}$$

where D is the absolute scalar.

The time integration of the evolution equation of ω or $\mathbf{v}$ is calculated by SOR method or rational Runge-Kutta method [11] with the second order accuracy in the time direction [see Appendix, (iv)]. All spatial derivatives are approximated by central differences except that the convective terms are approximated by the following upwind schemes:

In case (a), Greenspan scheme [12] with first order accuracy

$$F\partial G/\partial x=F(G_{I+1}-G_{I-1})/2\Delta x+|F|(-G_{I+1}+2G_I-G_{I-1})/2\Delta x,\tag{20}$$

and in case (b), Kawamura scheme [9] with third order accuracy

$$\begin{aligned}F\partial G/\partial x=&\,F(-G_{I+2}+8G_{I+1}-8G_{I-1}+G_{I-2})/12\,\Delta x\\&+|F|\,(G_{I+2}-4G_{I+1}+6G_I-4G_{I-1}+G_{I-2})/4\Delta x,\end{aligned}\tag{21}$$

are used.

4. Procedure of Numerical Simulation

The procedures of numerical simulation in both cases are similar, and are written as follows

(1) Generate stational grids by using the grid generation technique.

(2) Generate the grid at each time step by interpolation, and calculate the metrics and the velocities. In our simulations, the metrics: $\mathbf{e}_i$, g^{ij}, and $\left\{ {i \atop j\,k} \right\}$, and the grid velocity $\tilde{V}^i$ are needed, and are calculated using the following relations

$$\text{Natural basis } \mathbf{e}_i;\ \mathbf{e}_i = \partial \mathbf{z}/\partial x^i,$$

where $\mathbf{z}$ denotes the rectangular coordinates of a grid point, and x^i the general moving coordinates of the point.

Contravariant metric tensor g^{ij} is calculated by $g^{ij} g_{jk} = \delta^i_k$, and the metric tensor g_{ij} is given by the inner product of $\mathbf{e}_i$ and $\mathbf{e}_j$.

Riemann connection (Christoffel symbols of second kind) $\left\{ {i \atop j\,k} \right\}$;

$$\left\{ {i \atop j\,k} \right\} = g^{in}(\partial g_{jn}/\partial x^k + \partial g_{kn}/\partial x^j - \partial g_{jk}/\partial x^n)/2,$$

or the relation $\partial \mathbf{e}_i/\partial x^j = \left\{ {n \atop i\,j} \right\} \mathbf{e}_n$ is used.

Gird velocity $\tilde{V}^i$; $V = \tilde{V}^i \mathbf{e}_i = V^m i_m$ $(m = x, y)$, and V^x, V^y are calculated by the motion of grid. Solving the above equation we obtain

$$\tilde{V}^1 = V^x e^y_2 - V^y e^x_2, \qquad \tilde{V}^2 = -V^x e^y_1 + V^y e^x_1.$$

(3) Solve the Poisson's equation of ψ or P by SLOR method.

(4) Solve the evolution equation of ω or $\tilde{v}^i$ by SOR method or rational Runge-Kutta method.

The procedure of numerical simulation is illustrated as follows:

$$\begin{array}{ccccc} \text{START} \rightarrow (1) \rightarrow & t = t_0 & \rightarrow (2) \rightarrow & (3) \\ & \uparrow & & \downarrow \\ & t = t_0 + \Delta t & \leftarrow & (4) \end{array}$$

and the computations were carried out by the computer of FACOM-M380.

5. Numerical Examples and Results

a. *Blood Flow in Human Ventricle*

The first computation example is given for the numerical simulation of blood flow in human left ventricle. From two-dimensional contours taken by the cineangiography, the stational grids were generated by an algebraic method proposed by Gordon and Hall [13], see Appendix, (iii). In this numerical example the generation of the computational grids is one of the most difficult problems. As we have only obtained the outlines of the left ventricle, we do not know where is the fixed point during the motion of one cycle and where a point moves in the next figure etc. Therefore by observing the figures of left ventricle, we must determine the same point during one cycle, and in such case the algebraic method is convenient. The contour of left ventricle is parametrized to be the boundary of unit square $[0, 1] \times [0, 1]$ by using the interpolation of spline. Figure 1 shows the relation between the physical and the computational planes. The number of figures obtained is 32 per one cycle (1 sec), and 32 stational grids are generated. Figure 2 shows 8 stational grids in one cycle. In computing the grid velocities we must know the fixed point during the motion of body, and it is assumed that the middle point of apex-outlet (see Fig. 1) is fixed in space.

On the boundary conditions, the body and the computational grid fitted to the body move, and the velocity of body surface coincides with the velocity of grid; $\mathbf{v} = \mathbf{V}$ on the surface of body. From Eq. (15) the boundary values of the stream function are determined by putting $\psi = 0$ at the end point of a line and by integrating Eq. $(15)_1$ or $(15)_2$ along the boundary lines. The boundary condition of ω are determined by solving Eq. (17). On the outlet or inlet line the velocity of fluid is assumed to be constant and is calculated by the time-change of area of the body. The values of ψ and ω on the lines are determined in similar way to those of boundary surface, however, in this case the value of outer point is approximated by the extrapolation from the values at the neighbouring grid points just inside the outlet or inlet line.

Th computation starts at end diastole and our computational results in third cycle are presented in the following. The coefficient of viscosity of blood is assumed to be 0.03 poise, and the grid has 51×51 points, and the time increment in computation is chosen to be 0.0004. Figures 3 and 4 show the streamline and the velocity vectors during one cycle, respectively. Figure 5 shows the iso-vortex contours. Though our computation is carried for two-dimensional flow neglecting the existence of valves, the characteristic feature of velocities and pressure at the outlet

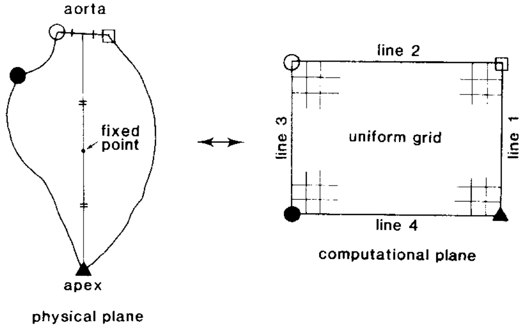

FIG. 1. Correspondence between physical plane and computational plane.

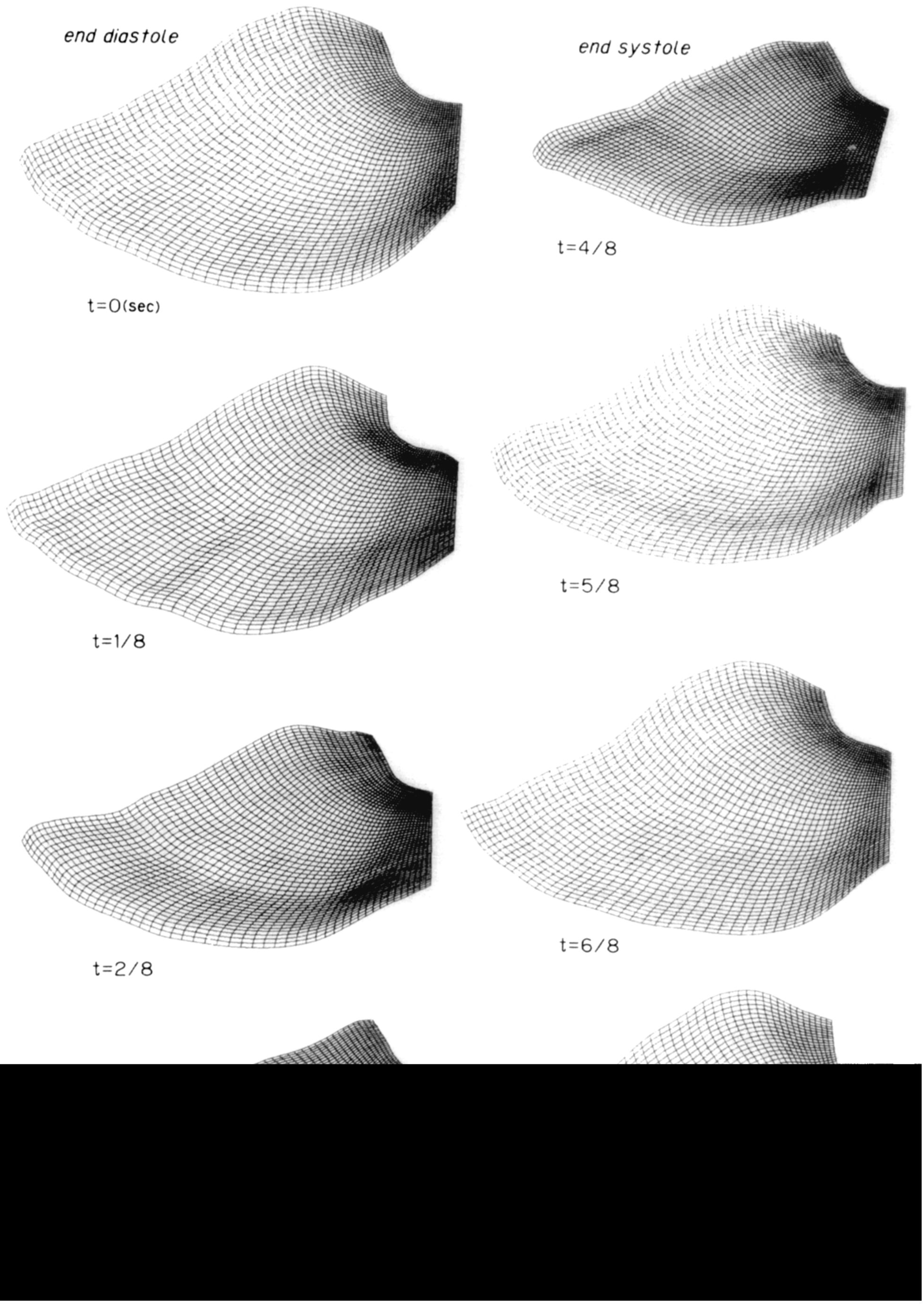

FIG. 2. Example of stational grids for left ventricle.

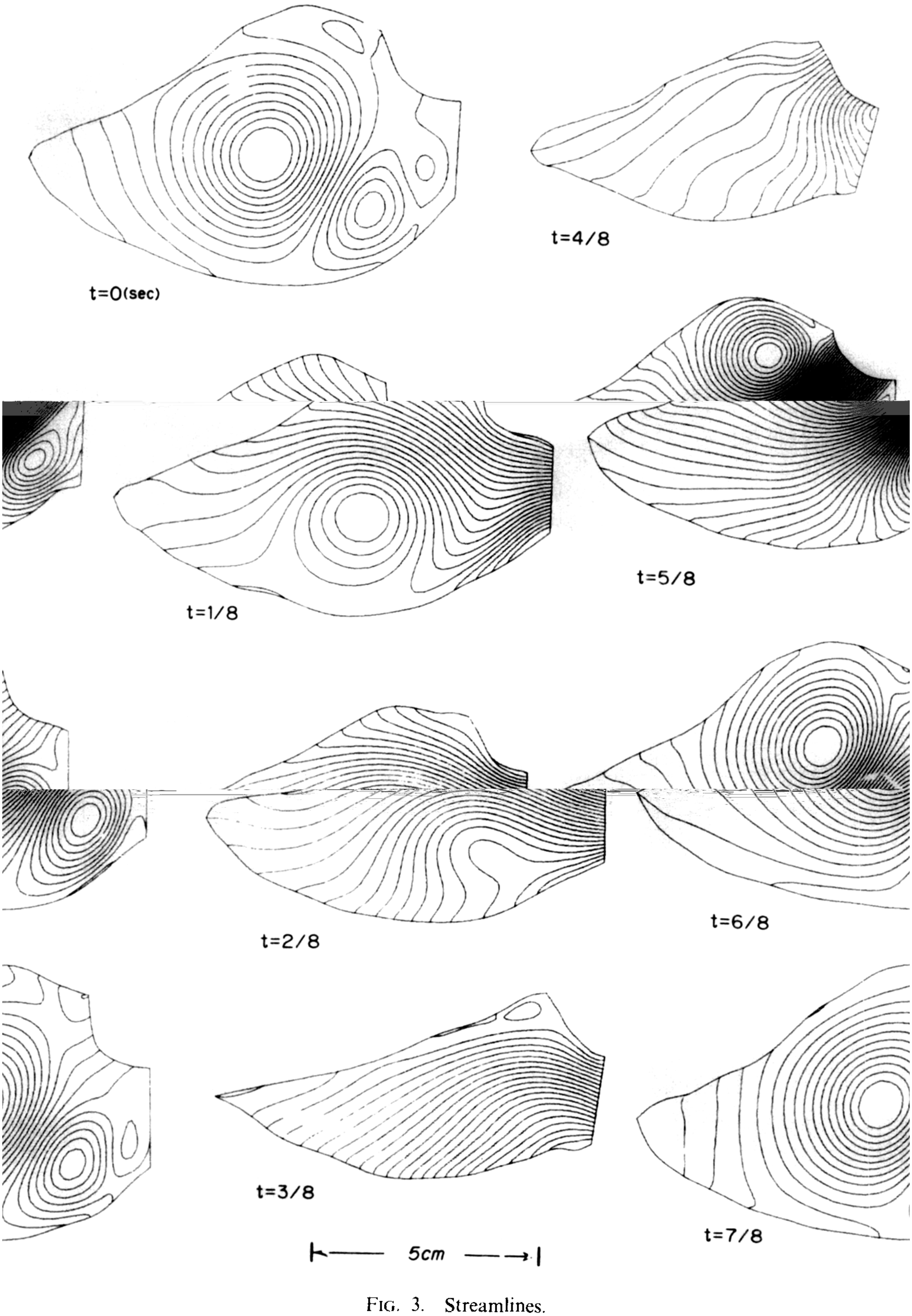

FIG. 3. Streamlines.

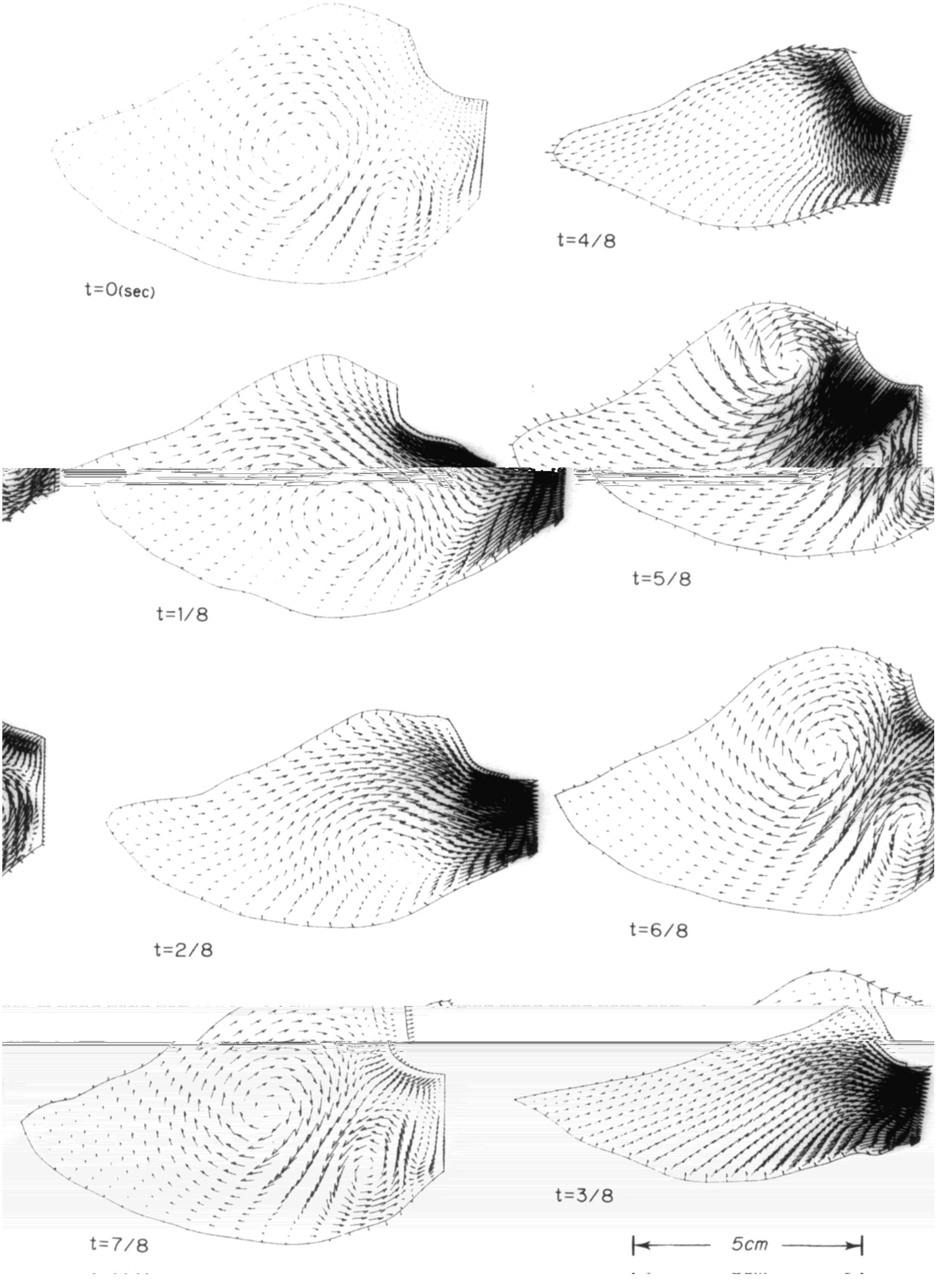

FIG. 4. Velocity vectors.

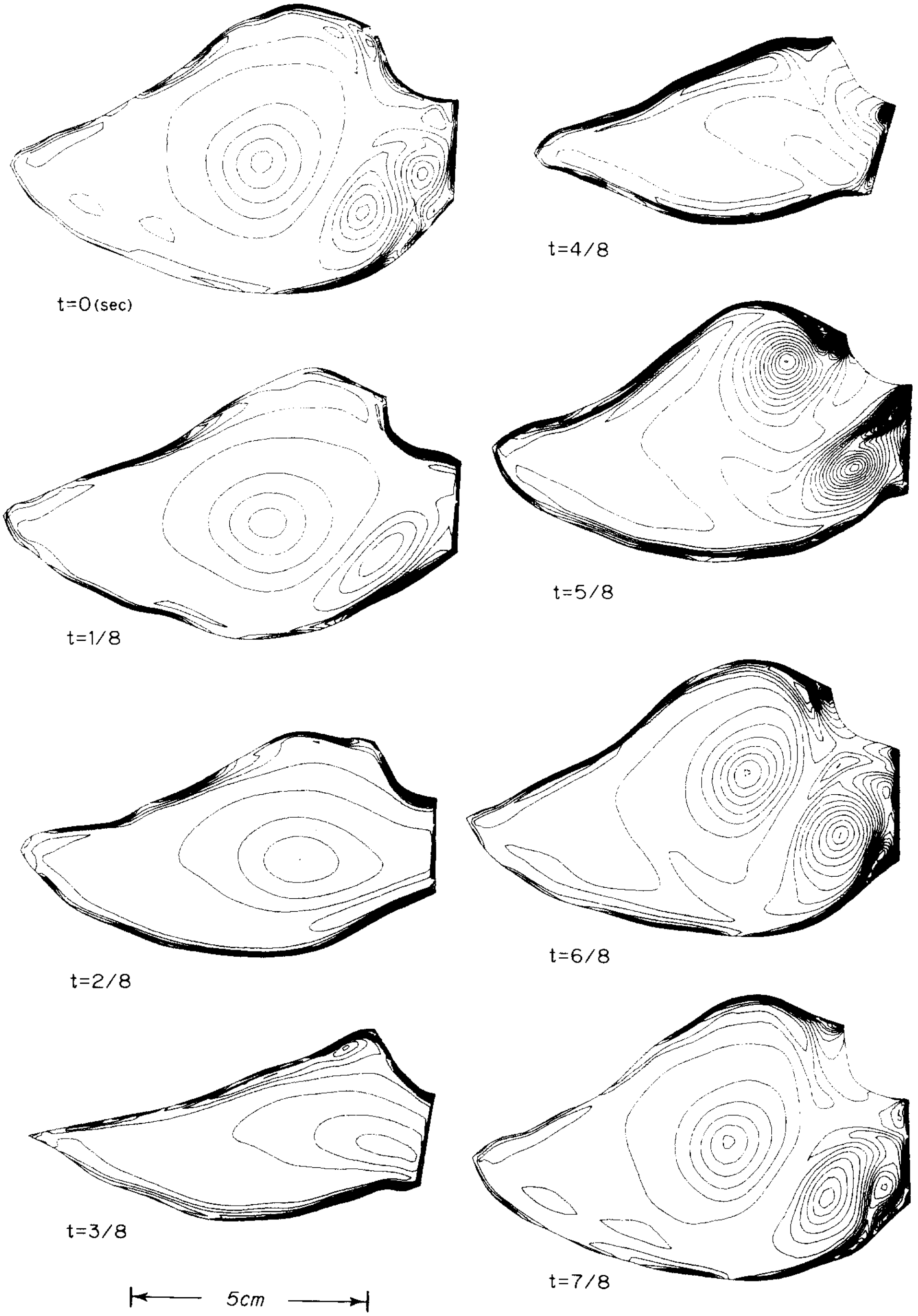

FIG. 5. Isovortex contours.

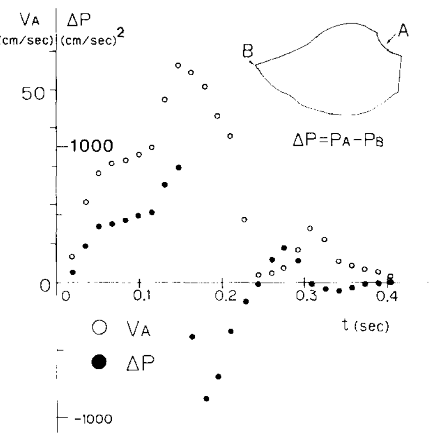

FIG. 6. Velocity and pressure in diastolic phase.

or inlet coincide with the experimental ones [14] (see Fig. 6). It is clear that two large swirls are observed during diastolic phase, and this observation has been confirmed experimentally.

There are very few data for human heart obtained by in vivo experiments, however, if three-dimensional data of the contour of heart is obtained, we will be able to compute the inner flow fields by the similar procedures of this paper.

b. *Dynamic Stall on Oscillating Airfoil*

The second computational example is given for the dynamic stall process of oscillating airfoil whose numerical simulations are the urgent problems in aerodynamics. In this case $P-\mathbf{v}$ variables are used, and Eqs. (9) and (12) are solved. The numerical experiment is given for NACA0012 airfoil oscillating in pitch at the amplitude of 5° about the mean angle of attack of 15° with the reduced frequency of 0.4π and the Reynolds number 20000. By the above algebraic method stational grids around the airfoil are first generated for three cases of its extreme angles of attack and its mean incident angle of attack. Then the grid at the arbitrary time step was calculated by the interpolation between two grids. The minimum spacing in the direction to the surface of the airfoil is fixed to be 0.0001 of the chord length. The grid has 171×61 points, and the 91 points are distributed on the airfoil surface, see Fig. 7.

On the boundary conditions, the nonslip condition is required for the viscous flow, and $\mathbf{v} = \mathbf{V}$ is set. The surface pressure is computed by solving the momentum equations on the surface of airfoil. For the far-field boundary, the quality of fluid is assumed to resemble the potential flow, and the following conditions are imposed: The normal component of velocity is zero except for the downstream one, which is extrapolated from the value at the neighbouring grid points just inside the boundary. The tangential component is so determined that the circulation coincides with

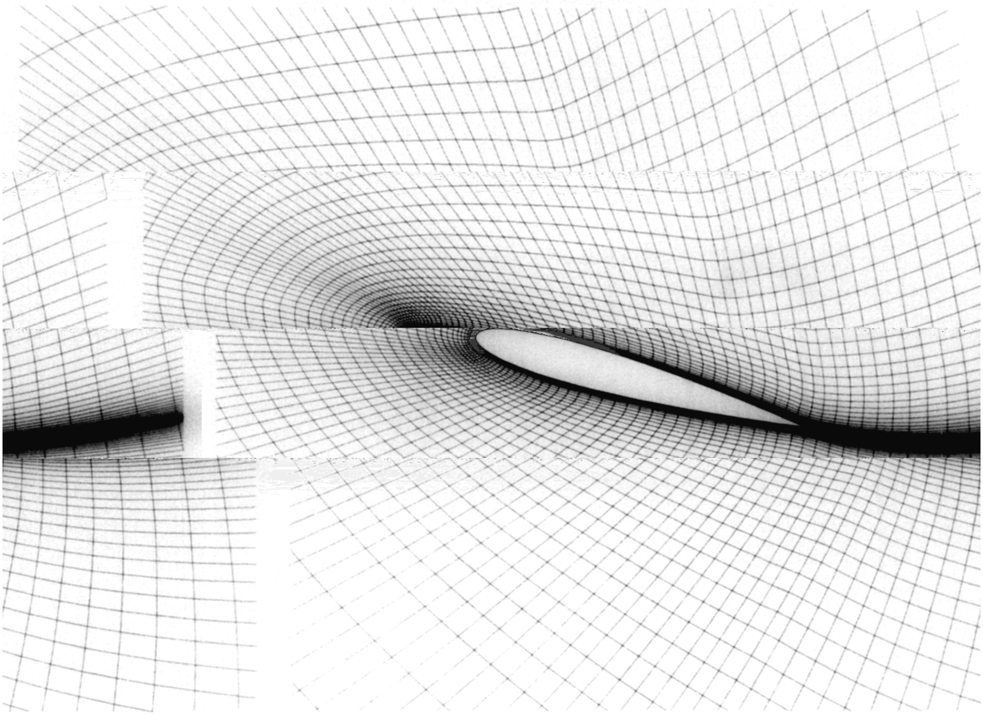

FIG. 7. Grid of NACA0012 airfoil, $\alpha = 15^\circ$.

that of inner grid points. The pressure on the outer boundary is set to be constant except for the downstream one, which is extrapolated in similar way to the normal component of velocity

The computation starts at the incident angle of 15° and our computational results in the third cycle are presented. Figure 8 shows the unsteady pressure distribution, and the suction peak gets higher after the incident angle exceeds the static stall one. At about $\alpha = 18^\circ$ the separation bubble is detected around the leading edge, and as the vortex converts itself along the upper airfoil surface, small bubbles are made up around the leading edge and there are many small pressure bumps on the upper surface. This is similar to the result of compressive laminar calculation by Ono [15]. Figure 9 shows the moment, drag and lift coefficients versus time. At $t = 0$, the incident angle is 15° and varies as $15^\circ \rightarrow 20^\circ \rightarrow 15^\circ \rightarrow 10^\circ \rightarrow 15^\circ$. The lift coefficient increases even after the incident angle exceeds the extreme angle of attack, and its reason, we consider, is that the reduced frequency is large and the effects of inertia are dominant in this case.

[illegible]

(Fig. 8) and we think the number of grid points on the airfoil surface is still small. Furthermore, three-dimensional computation will be needed if we want to capture the phenomena of separation clearly.

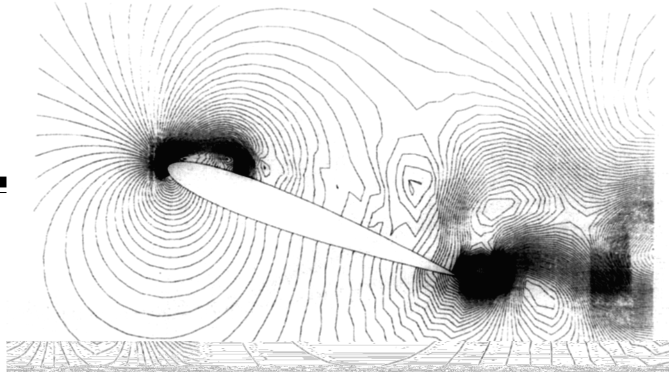

FIG. 8. Unsteady pressure distribution. (a) $t = 0.708$, $\alpha = 18.88°$, (b) $t = 1.195$, $\alpha = 19.98°$, (c) $t = 2.583$, $\alpha = 14.48°$, (d) $t = 3.114$, $\alpha = 11.15°$, (e) $t = 4.280$, $\alpha = 11.07°$.

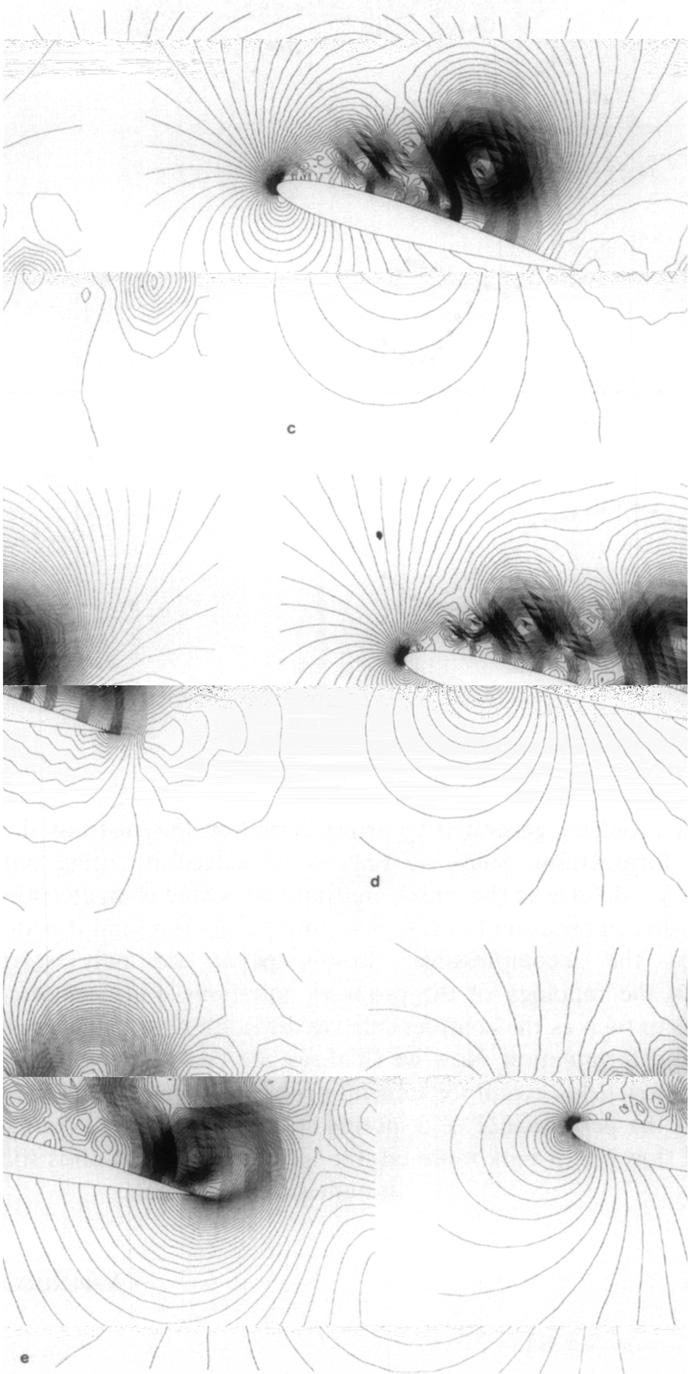

FIG. 8—*Continued.*

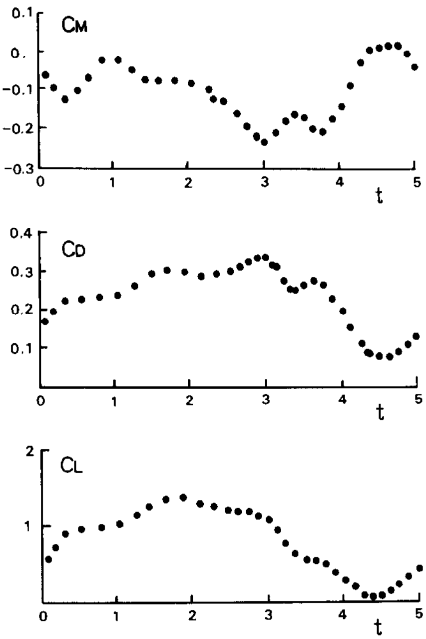

FIG. 9. Airloads.

6. Conclusion

It becomes clear that the unsteady flow problems generated by arbitrary motion of bodies can be solved by using our formulation. Since we only handle the geometric quantities whose characteristics are definite in the tensor analysis, the formulation is simple and clear, and its extensive application to other flow problems is easy. In this paper we only solved the incompressible two-dimensional Navier–Stokes equations. Furthermore, if the topology of the problem does not change during the motion, any flow problem such as the compressible three-dimensional equations can be solved by using our formulation. Now we think, there will be left few difficult problems in calculating these flow fields except for the memories and the computing speed of computer. Though the performance of computer has been advanced, the speed of computation is still slow and it took more than several hours for each numerical simulation.

Appendix

(i) *Lie Derivative* [4, 5]

At time $t = t_0$, let us consider two coordinate systems: x^i, the Euler coordinates fixed in space, and $\mathbf{x}^i_*$ the moving coordinates which coincide with x^i at time $t = t_0$.

The relation between two coordinates is

$$x^i = x^i(\mathbf{x}_*, t - t_0) \qquad \text{and} \qquad x^i = \mathbf{x}^i_* \quad \text{at } t = t_0.$$

Expanding the above equation by t

$$x^i = x^i + \Delta t V^i(\mathbf{x}_*, 0) + \cdots, \tag{A1}$$

where $V^i = (\partial x^i/\partial t)|\mathbf{x}_*$. Let $\mathbf{e}_i$ be the natural basis of Euler coordinates and $\mathbf{e}_i^*$ the natural basis of moving coordinates, then

$$\mathbf{e}_i^* = (\partial x^j/\partial \mathbf{x}^i_*)\, \mathbf{e}_j \tag{A2}$$

holds. Substituting Eq. (A1) into Eq. (A2), we obtain the following relation for the natural basis

$$\mathbf{e}_i^* = (\delta^j{}_i + \Delta t\, \nabla_i V^j)\, \mathbf{e}_i,$$

and for the covariant basis

$$\mathbf{e}^i_* = (\delta^i{}_j - \Delta t\, \nabla_j V^i)\, \mathbf{e}^j.$$

Using the above relations we can easily derive the evaluations of several quantities at the same coordinate in the following.

(a) Lie derivative of scalar density of weight w, $\tilde{Q}$.

Consider the scalar density field $\tilde{Q}$ of weight w, which satisfies the following relation. For Affine transformation $A^{i'}{}_j$: $dx^{i'} = A^{i'}{}_j\, dx^j$ (Fig. A),

$$\tilde{Q}' = \Lambda^{-W} \tilde{Q},$$

holds, where $\Lambda \equiv \mathrm{Det}(A^{i'}{}_j)$. At $(t_0 + \Delta t, P)$, the following relations between the Euler coordinates and the moving coordinates hold

$$dx^i = (\delta^i_j - \Delta t\, \nabla_j V^i)\, dx^j, \qquad \Lambda = \mathrm{Det}(\delta^i_j - \Delta t\, \nabla_j V^i) \doteqdot 1 - \Delta t\, \nabla_k V^k,$$

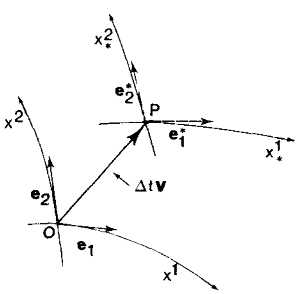

FIGURE A

and $\tilde{Q} + \Delta t\, \partial\tilde{Q}/\partial t^* = (1 + w\, \Delta t\, \nabla_k V^k)[\tilde{Q} + \Delta t(\partial\tilde{Q}/\partial t + V^k \nabla_k \tilde{Q})]$, where $\partial/\partial t^* \equiv \partial/\partial t|\mathbf{x}_*$. Comparing the first order terms of Δt, the Lie derivative of scalar density of weight w is derived as

$$\partial\tilde{Q}/\partial t^* = \partial\tilde{Q}/\partial t + V^k \nabla_k \tilde{Q} + w\tilde{Q}\, \nabla_k V^k, \tag{A3}$$

Note that $\sqrt{g} = [\mathrm{Det}(g_{ij})]^{1/2}$ is the scalar density of weight $+1$, and the Lie derivative of $\sqrt{g}$ is

$$\partial\sqrt{g}/\partial t^* = \partial\sqrt{g}/\partial t + \sqrt{g}\, \nabla_k V^k,$$

where $\nabla_k \sqrt{g} = 0$ used.

(b) Lie derivative of contravariant vector, F^i.

Let F^i be the contravariant vector field. It is expressed in Euler coordinates at $(t_0 + \Delta t, P)$ as $[F^i + \Delta t(\partial F^i/\partial t + V^k \nabla_k F^i)]\, \mathbf{e}_i$. In moving coordinates it becomes $(F^i + \Delta t\, \partial F^i/\partial t^*)\, \mathbf{e}_i^*$, and since both vectors are equal, we obtain

$$\partial F^i/\partial t^* = \partial F^i/\partial t + V^k \nabla_k F^i - F^k \nabla_k V^i. \tag{A4}$$

For the contravariant vector density of weight w, we obtain from Eqs. (A3) and (A4)

$$\partial\tilde{F}^i/\partial t^* = \partial\tilde{F}^i/\partial t + V^k \nabla_k \tilde{F}^i - \tilde{F}^k \nabla_k V^i + w\tilde{F}^i \nabla_k V^k. \tag{A5}$$

Using the similar procedures the Lie derivative of any tensor can be obtained.

(c) Lie derivative of covariant vector, F_i

$$\partial F_i/\partial t^* = \partial F_i/\partial t + V^k \nabla_k F_i + F_k \nabla_i V^k. \tag{A6}$$

(d) Lie derivative of tensor density of weight w, $\tilde{P}^i{}_j$,

$$\partial\tilde{P}^i{}_j/\partial t^* = \partial\tilde{P}^i{}_j/\partial t + V^k \nabla_k \tilde{P}^i{}_j - \tilde{P}^k{}_j \nabla_k V^i + \tilde{P}^i{}_k \nabla_j V^k + w\tilde{P}^i{}_j \nabla_k V^k. \tag{A7}$$

Note that the above relations hold in any time though we have derived the relations at the special time t_0.

(ii) *Viviand's Conservation Forms* [1, 3]

Equation (2) is equivalent to

$$\partial(\tilde{\rho}\mathbf{v})/\partial t + \nabla_k(\tilde{\rho}\mathbf{v}v^k - \tilde{\boldsymbol{\sigma}}^k) = \tilde{\rho}\mathbf{F},$$

where $\boldsymbol{\sigma}^k \mathbf{e}_k = \sigma^{ik}\mathbf{e}_i \mathbf{e}_k$, and since the Lie derivative of $\tilde{\rho}\mathbf{v}$ is the same as that of the scalar density of weight $+1$;

$$\begin{aligned}\partial(\tilde{\rho}\mathbf{v})/\partial t &= \partial(\tilde{\rho}\mathbf{v})/\partial t + V^k \nabla_k(\tilde{\rho}\mathbf{v}) + (\tilde{\rho}\mathbf{v})\, \nabla_k V^k \\ &= \partial(\tilde{\rho}\mathbf{v})/\partial t + \nabla_k(\tilde{\rho}\mathbf{v}V^k),\end{aligned}$$

we obtain

$$\partial(\tilde{\rho}\mathbf{v})/\partial t + \nabla_k[\tilde{\rho}\mathbf{v}(v^k - V^k) - \tilde{\boldsymbol{\sigma}}^k] = \tilde{\rho}\mathbf{F}.$$

If $\mathbf{v}$ is expressed by the rectangular base vector $\mathbf{i}_m$

$$\mathbf{v} = v^m \mathbf{i}_m \qquad (m = x, y, z),$$

then let the base vector $\mathbf{i}_m$ be out of the derivatives, and using the following relation for the contravariant vector density of weight $+1$;

$$\nabla_k \tilde{J}^k = \partial_k \tilde{J}^k,$$

Eq. (8) is obtained. The same equation is derived from Eq. (4), by putting the natural basis $\mathbf{e}_i$ in the derivative.

(iii) *Grid Generation* [13]

The algebraic grid generation procedure proposed by Gordon and Hall is as follows:

In two-dimensional case, let $\mathbf{F}$ be the continuous function such that $\mathbf{F}$: $\partial\Phi \to \partial R^2$, where Φ is $[0, 1] \times [0, 1]$, then the univalent function $\mathbf{U}$: $\Phi \to R^2$ is constructed from $\mathbf{U} = Ps \oplus Pt)[\mathbf{F}]$. The projector $Ps \oplus Pt$ is defined by

$$Ps \oplus Pt[\mathbf{F}] = \phi_0(s)\,\mathbf{F}(0, t) + \phi_1(s)\,\mathbf{F}(1, t) + \chi_0(t)\,\mathbf{F}(s, 0) + \chi_1(s)\,\mathbf{F}(s, 1)$$
$$- \sum_{i=0}^{1} \sum_{j=0}^{1} \phi_i(s)\,\chi_j(t)\,\mathbf{F}(s_i, t_j),$$

where ϕ_0, ϕ_1, and χ_0, χ_1 are four blending functions which satisfy the cardinality conditions

$$\phi_i(s_k) = \delta_{ik}, \qquad \chi_i(t_k) = \delta_{ik} \qquad \text{for} \quad i, k = 0, 1$$

where $s_0 = t_0 = 0$ and $s_1 = t_1 = 1$.

(iv) *Wambecq's Rational Runge-Kutta Method for Solving System of Ordinary Differential Equations* [11].

Suppose the system of differential equations $d\mathbf{w}/dt = \mathbf{f}(\mathbf{w})$ is given together with the initial values $\mathbf{w}(t_0) = \mathbf{w}_0$, where

$$\mathbf{w} = (w_1, w_2, \ldots, w_N)^{\mathrm{T}}, \qquad \mathbf{f}(\mathbf{w}) = [f_1(\mathbf{w}), f_2(\mathbf{w}), \ldots, f_N(\mathbf{w})]^{\mathrm{T}},$$

and $\mathbf{w}_0 = (w_{01}, w_{02}, \ldots, w_{0N})^{\mathrm{T}}$ are elements of R^N. An approximation $\mathbf{w}$ to the value of $\mathbf{w}$ at the point $t = t_0 + \Delta t$ is obtained as the following:

$$\mathbf{w} = \mathbf{w}_0 + [2\mathbf{g}^1(\mathbf{g}^1 \cdot \mathbf{g}) - \mathbf{g}(\mathbf{g}^1 \cdot \mathbf{g}^1)]/(\mathbf{g} \cdot \mathbf{g}),$$

where $\mathbf{g}^1 \equiv \Delta t\mathbf{f}(\mathbf{w}_0)$, $\mathbf{g}^2 = \Delta t\mathbf{f}(\mathbf{w}_0 + a\mathbf{g}^1)$, $\mathbf{g} \equiv (1-b)\,\mathbf{g}^1 + b\mathbf{g}^2$, $ab \leqslant -\frac{1}{2}$, and $\cdot$ symbolizes an inner product of two vectors. The method gives the second order accuracy if $ab = -\frac{1}{2}$, and the first order accuracy if $ab < -\frac{1}{2}$.

ACKNOWLEDGMENTS

The authors would like to express their sincere gratitude to Hajime Miyoshi, Professor Motoaki Sugawara, and Professor Kunio Kuwahara for many useful discussions and suggestions. They also express their thanks to Professor Nobunori Oshima for his guidance of tensor analysis.

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